AN ANALYSIS OF

AN ARCH SUPPORTED CONTINUOUS GIRDER

by

G. Ford and R.H. McManus

Thesis

An Analysis of
An Arch Supported Continuous Girder

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George Ford
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Ralph N. McManus.

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I. F. Morrison.

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Diagrams

Introduction

The arch supported continous girder, more commonly known as the "Tied Arch", somewhat resembles the ordinary arch bridge so widely seen today. However the fact that this structure has a lower chord that will carry moment, thrust and shear, enables the engineer to distribute the moment to both the rib and the girder. This at once allows the use of heavier members in the chord region and lighter rib members which is, in itself, an obvious advantage in construction. The rib and girder, while being stable within themselves, require only light hangers to distribute the moment. Then too, the elimination of the diagonals (so necessary to a truss bridge) adds to its aesthetic qualities.

This type of structure, although fairly common in Europe, is relatively new to this continent. There are consequently few methods of analysis available in our literature. The general approach has been to use differential calculus with necessary assumptions and approximations. European literature, on the other hand, indicates that the calculus of finite differences offers a powerful approach to such an analysis. This is especially true in the case where concentrated loads occur at regular intervals. Accordingly, then, an attempt will be made to obtain a general analysis of the tied arch by means of difference equations.

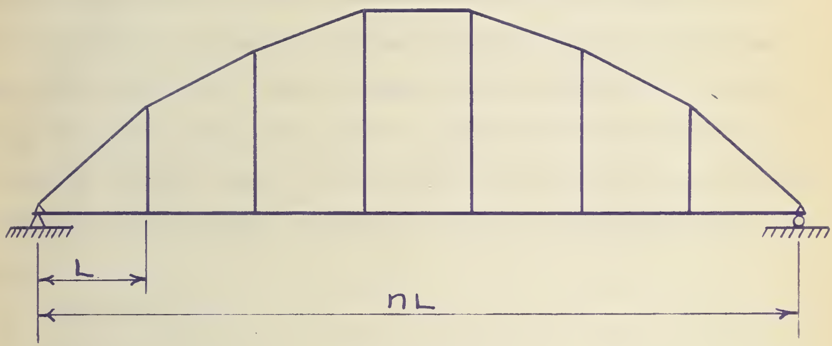


FIGURE 1

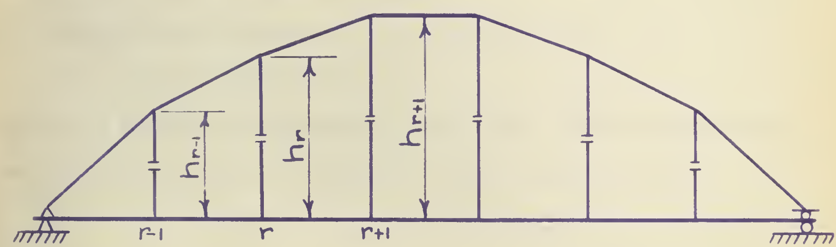


FIGURE 2

THE STATICALLY DETERMINATE STRUCTURE.

Type of Structure

The structure to be dealt with (Figure 1,) will consist of an arched rib held together by a tie girder which, in turn, is supported at regular intervals by vertical hangers connected to the rib at the panel points. For the purpose of analysis, the girder and rib will be considered to be pin connected and the structure, as a whole, to be simply supported. Both the rib and the girder can take moment, shear and thrust, while the hangers will be restricted to tensile stress only. In order to have a load system of concentrated forces occuring at regular intervals the floor beams will frame into the girder at the panel points.

With the use of the following notation the degree of indeterminateness may be obtained:

Let n = number of panels

k = number of joints = $2n$

r_1 = number of stiff members = $2n$

r_2 = number of simple members = $(n-1)$

e = number of stiff corners = $2(n-1)$

Degree of indeterminateness is $r_1 + r_2 + e - 2k + 3 = N$

$$N = 2n + (n-1) + 2(n-1) - 4n + 3 = n$$

The system is therefore indeterminant to nth degree. If the connections between the girder and rib are not pinned then $e = 2n$ and $N = n + 2$.

Assumptions

The computations will be carried out on the basis of the following assumptions:

1. All members will be considered straight between panel points.
2. The loads are vertical and may come on the structure only at panel points.
3. The moments of inertia of all members between panel points will be constant and will be so chosen that the relation $\dfrac{I_r^R \operatorname{Cos} \phi_r}{I_r^G} = \alpha$ will be constant throughout the structure.
4. The hangers will be slender pin connected members and will be restricted to tensile stresses only.
5. The bottom chord will be horizontal.
6. The deformations due to normal and shearing stresses will be neglected throughout the structure.

Notation

A = Area of Cross section.

A_r^G = area of the rth member in the girder.

A_r^R = area of the rth member in the rib.

E = Modulus of elasticity (assumed constant for the entire structure).

H = The horizontal component of rib compression or tie tension.

I = Moment of Inertia.

I_r^R = the moment of inertia of the rth member in the rib.

I_r^G = the moment of inertia of the rth member in the girder.

ϕ_r = The slope of the arch rib in the rth panel.

h_r = The rise of the arch rib at the rth panel point.

L = The length of the panel.

M_r' = The simple beam moment at the rth panel point.

M_r^t = The total bending moment to be resisted by the girder and tie at the rth panel point = $(M_r' - Hh_r)$.

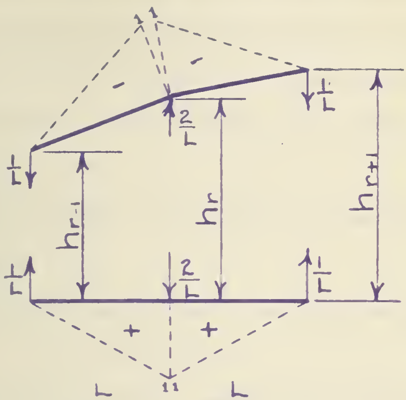
M_r^G = The moment to be resisted by the girder at the rth panel point.

M_r^R = The moment to be resisted by the rib at the rth panel point.

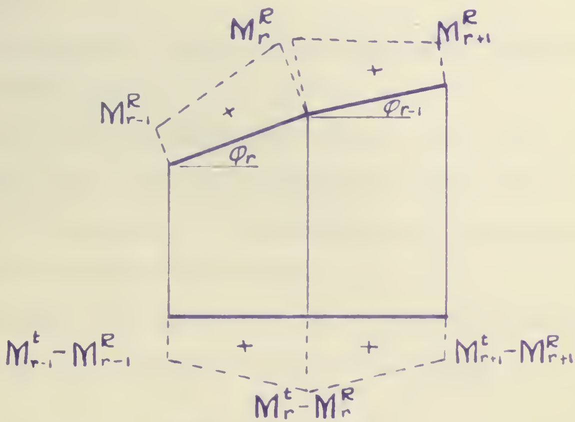
$\overline{M}$ = The moment due to a unit couple.

$\overline{N}$ = The direct stress in any member of the structure due to $H = 1$.

$$\alpha = \frac{I_r^R \cos \phi_r}{I_r^G}$$



$\overline{M}$ SURFACE
FIGURE 3



M SURFACE
FIGURE 4

Analysis

The necessary and sufficient conditions for determining the unknowns in a structure, indeterminate to the nth degree, are that $n+3$ independent equations be obtained. In general the n equations (in addition to the three equations of statics) are most satisfactorily obtained from the work equations. These work equations are obtained from the law of virtual work and may be expressed algebraically as:

$\theta = \sum \int \frac{\bar{M}\,M\,dx}{EI}$ = The change in angle at the point of application of the unit couple.

$\delta = \sum \int \frac{\bar{M}\,M\,dx}{EI} + \sum \int \frac{\bar{N}\,N}{A\,E}\,dx$ = The deflection at the point of application of the unit load.

The $\int \frac{\bar{M}\,M}{E\,I}\,dx$ may be evaluated by the moment area method (see appendix A) or by the Muller-Breslau tables. (3)

The analysis resolves itself into the solution of two main problems:

(a) The determination of the division of the total moment between the rib and the girder.

(b) The determination of the horizontal components of the longitudinal forces in the chord members.

The computation for (a) takes place under the use of the first of the work equations. By placing a unit couple at each panel point successively an $\bar{M}$ surface, Figure 3, is obtained which combined with the M surface of the structure, Figure 4, gives rise to a three term equation as follows:

Since M_r^R is the moment to be resisted by the rth panel of the rib,

Then $M_r^t - M_r^R$ is the moment in the rth panel of the girder.

(analysis cont.)

θ for the top chord at the rth panel $= \sum \int \frac{\bar{M}\ M}{E\ I}\, dx$

$$= \frac{L}{6EI_r^R \text{ Cos } \phi_r}\left[M_{r-1}^R + 2\ M_r^R\right]$$

θ for the lower chord at the rth panel

$$= \frac{L}{6EI_r^G}\left[M_{r-1}^t - M_{r-1}^R + 2(M_r^t - M_r^R)\right]$$

Since the hangers do not change in length, then the change in the angle at the joints in the upper and lower chords must be equal.

Therefore, $\frac{L}{6EI_r^R \text{ Cos } \phi_r}\left[M_{r-1}^R + 2M_r^R\right] = \frac{L}{6EI_r^G}\left[M_{r-1}^t - M_{r-1}^R + 2M_r^t - 2M_r^R\right]$

Therefore, $\left(M_{r-1}^R + 2M_r^R\right)\ (1 + \alpha) = (M_{r-1}^t + 2M_r^t)$

Therefore, $(M_{r-1}^R + 2M_r^R) = \frac{\alpha}{1+\alpha}\ (M_{r-1}^t + 2M_r^t)$

Similarly in the r + 1 panel we have:

$$(2M_r^R + M_{r+1}^R) = \frac{\alpha}{1+\alpha}\ (2M_r^t + M_{r+1}^t)$$

Adding we have,

$$M_{r-1}^R + 4M_r^R + M_{r+1}^R = \frac{\alpha}{1+\alpha}\ (M_{r-1}^t + 4M_r^t + M_{r+1}^t)$$

The general solution of this difference equation is:

$$M_r^R = C_1\ K_1^r + C_2\ K_2^r + \frac{\alpha}{1+\alpha}\ (M_r^t)$$

Where K_1 and K_2 are the roots of the characteristic equation

$$K^2 + 4K + 1 = 0$$

and C_1 and C_2 are constants to be determined from the boundary conditions.

Also since $K_1\ K_2 = 1$ we may write $M_r^R = C_1\ K^r + C_2\ K^{-r} + \frac{\alpha}{1+\alpha}\ M_r^t$

(analysis cont.)

and since $M_0^t = M_n^t = 0$ and $h_0 = h_n = 0$

Then $C_1 = C_2 = 0$

The general solution now becomes

$$M_r^R = \frac{\alpha}{1+\alpha} \quad M_r^t$$

$$M_r^G = M_r^t - M_r^R$$

$$= M_r^t - \frac{\alpha}{1+\alpha} \quad M_r^t$$

$$= \frac{1}{1+\alpha} \quad M_r^t$$

Therefore, $\dfrac{M_r^G}{M_r^R} = \dfrac{1}{\alpha}$

It will be well to note here that the above solution may be obtained by considering the two term equation developed for the change in angle at the rth panel point.

Then $M_{r-1}^R + 2M_r^R = \dfrac{\alpha}{1+\alpha} \, (M_{r-1}^t + 2M_r^t)$

A particular solution is

$$M_r^R = \frac{\alpha}{1+\alpha} \quad M_r^t$$

The solution of the homogenious equation

$$M_{r-1}^R + 2M_r^R = 0$$

is $\quad M_r^R = -\frac{1}{2} M_{r-1}^R$

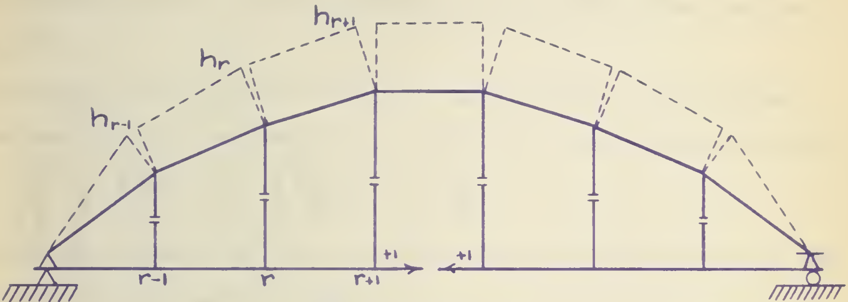
$$= (-\tfrac{1}{2})^r \, M_0^R$$

The general solution becomes;

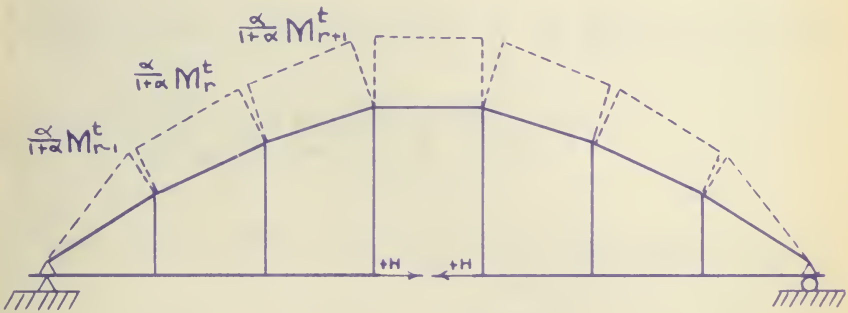
$$M_r^R = \frac{\alpha}{1+\alpha} \quad M_r^t + (-\tfrac{1}{2})^r \, M_0^R$$

But $M_0^R = 0$

Therefore $M_r^R = \dfrac{\alpha}{1+\alpha} \quad M_r^t$



$\overline{M}$ SURFACE
FIGURE 5



M SURFACE
FIGURE 6

(analysis cont.)

A complete solution now requires only that the horizontal component (H) be found. Since the hangers are assumed pin connected, it is obvious that the horizontal component (H) is constant. The second work equation allows the computation of H.

By placing a unit load in the girder an $\overline{M}$ surface (Figure 5) is obtained which combined with the M surface (Figure 6) gives rise to the following equation:

$$\delta = \sum_{0}^{n} \int_{0}^{\ell} \frac{\overline{M}\,M}{EI}\,dx + \sum_{0}^{n} \int_{0}^{\ell} \frac{\overline{N}\,N}{AE}\,dx = \sum_{0}^{n} \frac{HL}{A_G E}$$

By dividing the moment as above and substituting in the general equation the solution for H is determined as follows:

$$\sum_{0}^{n} \frac{\alpha}{1+\alpha} \left\{ \frac{h_{r-1}}{6I_r^R \operatorname{Cos} \phi_r} \left[2(M'_{r-1} - H\,h_{r-1}) + (M'_r - H\,h_r) \right] \right.$$

$$\left. + \frac{h_r}{6I_r^R \operatorname{Cos} \phi_r} \left[(M'_{r-1} - H\,h_{r-1}) + 2(M'_r - H\,h_r) \right] \right\} + \sum_{0}^{n} \frac{H}{\operatorname{Cos}^3 \phi_{r-1}\, A_r^R} = H \sum_{0}^{n} \frac{1}{A_R^G}$$

$$\sum_{0}^{n} \left[(2h_{r-1} + h_r)\, M'_{r-1} + (2h_r + h_{r-1})\, M'_r \right] - \sum_{0}^{n} 2H\,(h_{r-1}^2 + h_{r-1}\,h_r + h_r^2)$$

$$+ \frac{\alpha+1}{\alpha} H \sum_{0}^{n} \frac{6\,I_r^R}{\operatorname{Cos}^2 \phi_r\, A_r^R} = \frac{\alpha+1}{\alpha} H \sum_{0}^{n} \frac{6\,I_r^R \operatorname{Cos} \phi_r}{A_r^G}$$

$$\sum_{0}^{n} (2\,h_{r-1} + h_r)\, M'_{r-1} + (2\,h_r + h_{r-1})\, M'_r \Big] = 2\,H \sum_{0}^{n} (h_{r-1}^2 + h_{r-1}\,h_r + h_r^2)$$

$$- \frac{\alpha+1}{\alpha} H \sum_{0}^{n} \frac{6\,I_r^R}{\operatorname{Cos}^2 \phi_r\, A_r^R} + \frac{\alpha+1}{\alpha} H \sum_{0}^{n} \frac{6 \operatorname{Cos} \phi_r}{A_r^G}$$

Since the last two terms are small when compared to the first term, they can be neglected.

(analysis cont.)

Therefore;

$$H = \frac{\sum_{0}^{n} \left[(2\,h_{r-1} + h_r)\,M'_{r-1} + (2\,h_r + h_{r-1})\,M'_r\right]}{2\sum_{0}^{n} \left[(h^2_{r-1} + h_{r-1}\,h_r + h_r^{\,2})\right]}$$

Comparison of Results

J. M. Garrelts, (1) Associate member of the American Society of Civil Engineers, has presented an analysis of the Tied Arch, using the method of differential calculus. The Authors' derived equations for M_r^G and M_r^R check precisely with those derived by Professor Garrelts. Using the dimensions for the St. George's Tied Arch the equation for the determination of H gives a close numerical check on Professor Garrelts' values as shown in the following table.

Unit Load at Panel Point	H Garrelts'	H Authors'	% Difference
1	0.2358	0.2397	2.1
2	0.4595	0.4675	1.7
3	0.6625	0.6733	1.6
4	0.8372	0.8482	1.3
5	0.9772	0.9852	0.8
6	1.0772	1.0797	0.3
7	1.1300	1.1276	0.2

The differences show a steady decrease from a maximum of 2% with the unit load at panel point No. 1 to 0.2% with the unit load at the mid-panel point.

Conclusions

The solution of the following equations resolves the structure into a statically determinate case.

$$M_r^t = M_r' - H\,h_r$$

$$M_r^G = \frac{1}{1+\alpha}\quad M_r^t$$

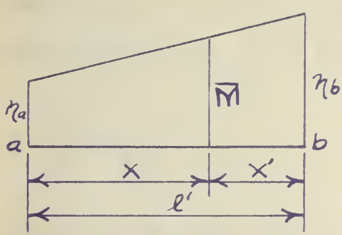
$$M_r^R = \frac{\alpha}{1+\alpha}\quad M_r^t$$

$$H = \frac{\sum_0^n \left[(2h_{r-1} + h_r)\,M'_{r-1} + (2h_r + h_{r-1})\,M'_r \right]}{2\sum_0^n \left[h^2_{r-1} + h_{r-1}\,h_r + h^2_r \right]}$$

All the necessary information for solving these equations may be obtained from the dimensions and loads for any particular Tied Arch. From the comparison of results with those of J. M. Garrelts it is to be noted that the computation for H is not appreciably affected by a small vertical curve in the girder. In this case, however, the h_r's are maintained as the vertical distance between upper and lower panel points.

With the moments at the panel points, and the horizontal in the girder and rib known, the stresses in the hangers may be found by the ordinary methods of statics.

Since the structure as a whole is simply supported, we may expect no temperature stresses.



$\overline{M}$ SURFACE

FIGURE 8

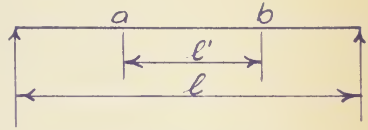
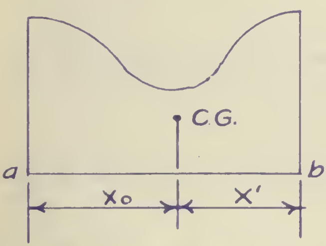
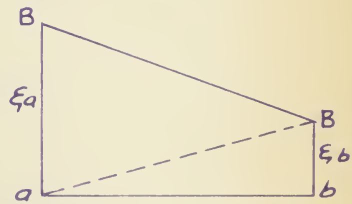


FIGURE 7



M SURFACE

FIGURE 9



M SURFACE

AS A TRAPAZOID

FIGURE 10

Appendix A

Consider a straight beam with o-x axis along the long axis of the beam. (Figure 7). Often a beam may not have a constant moment of inertia. When this is so, we may multiply both sides of equation by any constant moment of inertia I_0.

The work equation $EI\delta = \int_0^{\ell} \bar{M} M \, dx$ may be written

$$EI_0\delta = \int_0^{\ell} \bar{M} M \frac{I_0}{I} dx$$

The integral may be split into two parts.

Consider any section of beam a b

$\bar{M}$ (ordinate) $= \frac{x'}{\ell_1}\eta_a + \frac{x}{\ell_1}\eta_b$ (Figure 8)

$M\frac{I_0}{I}$ = distorted M surface. (See Figure 9)

$$\int_a^b \bar{M} M \frac{I_0}{I} dx = \frac{\eta_a}{\ell_1}\int_a^b M\frac{I_0}{I} x' dx' + \frac{\eta_b}{\ell_1}\int_a^b M\frac{I_0}{I} x \, dx.$$

$$\int_a^b M\frac{I_0}{I} x \, dx = S_{Ba} = \text{Statical moment.}$$

$$\int_a^b M\frac{I_0}{I} x' \, dx' = S_{Bb} = \text{Statical moment.}$$

$$E I_0 \delta = \frac{\eta_a}{\ell_1} S_{Bb} + \frac{\eta_b}{\ell_1} S B_a$$

As a special case consider the distorted M surface as a trapazoid. (Figure 10).

(Appendix A cont.)

$$S_{Ba} = \frac{\xi_a \ell_1^2}{6} + \frac{\xi_b \ell_1^2}{3}$$

$$S_{Bb} = \frac{\xi_a \ell_1^2}{3} + \frac{\xi_b \ell_1^2}{6}$$

$$EI_0 \delta_m = \frac{\eta_a \ell_1}{6} (2\xi_a + \xi_b) + \frac{\eta_b \ell_1}{6} (\xi_a + 2\xi_b)$$

Appendix B.

The computation for H becomes somewhat simplified if one takes advantage of the symmetry of the structure.

As a numerical example consider the fifteen panel Tied Arch analysed by Professor Garrelts, from which we have the following.

$n = 15$ panels.

$L = 35.958$ feet per panel.

Table 1

r	h_r ft.	$2h_r$ ft.
0	0	0
1	23.213	46.426
2	43.109	86.218
3	59.690	119.380
4	72.954	145.908
5	82.903	165.806
6	89.535	179.070
7	92.851	185.702
8	92.851	185.702
9	89.535	179.070
10	82.903	165.806
11	72.954	145.908
12	59.690	119.380
13	43.109	86.218
14	23.213	46.426
15	0	0

(Appendix B cont.)

In the equation $H = \dfrac{\sum_{0}^{n}\left[(2h_{r-1} + h_r)\, M'_{r-1} + (2h_r + h_{r-1})\, M'_r\right]}{2\sum_{0}^{n}\left[h^2{}_{r-1} + h_{r-1}\, h_r + h_r^2\right]}$

It is evident that one computation of the denominator is sufficient for all points on the influence line. Also, we note that the sum over the structure is twice the sum for 7 panels plus the eighth panel. In summing the numerator it is to be noted that the only variation, with the unit load at the different panel points, is in the M'_r.

By summing the two columns in table 1, in the order indicated by the arrows, we arrive at the constant coefficients for the M'_r.

Thus: $\sum_{0}^{n}\left[(2h_{r-1} + h_r)\, M'_{r-1} + (2h_r + h_{r-1})\, M'_r\right]$

$$= 0\,(23.213) + M'_1\,(46.426) + M'_1\,(89.535) + M'_2\,(109.431)$$
$$+ M'_2\,(145.908) + \text{---}$$
$$+ M'_{13}\,(145.908) + M'_{13}\,(109.431) + M'_{14}\,(89.535) + M'_{14}\,(46.426)$$
$$+ 0\,(23.213).$$
$$= (M'_1 + M'_{14})\,(46.426 + 89.535) + (M'_2 + M'_{13})\,(109.431 + 145.908)$$
$$+ \text{---}$$
$$\text{--------------} + (M'_7 + M'_8)\,(275.237 + 278.553).$$

With the unit load at any panel point the M'_1, M'_2 etc. may be easily computed and the seven values summed to give the numerator in the H equation.

As the unit load moves across the structure the values $(M'_1 + M'_{14})$, $(M'_2 + M'_{13})$ etc. are repetitive up to the point of application of the unit load. Thus subtotals for the computation with the unit load at panel point r may be used for

(Appendix B cont.)

that with the unit load at panel point r + 1:

Table 2

Unit Load panel pt.	$M'_1 + M'_{14}$	$M'_2 + M'_{13}$	$M'_3 + M'_{12}$	$M'_4 + M'_{11}$	$M'_5 + M'_{10}$	$M'_6 + M'_9$	$M'_7 + M'_8$
1	$L \times \frac{15}{15}$	$L \times \frac{15}{15}$	$L \times \frac{15}{15}$	$L \times \frac{15}{15}$	$L \times \frac{15}{15}$	$L \times \frac{15}{15}$	$L \times \frac{15}{15}$
2	$L \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$
3	$L \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 3 \times \frac{15}{15}$	$L \times 3 \times \frac{15}{15}$	$L \times 3 \times \frac{15}{15}$	$L \times 3 \times \frac{15}{15}$	$L \times 3 \times \frac{15}{15}$
4	$L \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 3 \times \frac{15}{15}$	$L \times 4 \times \frac{15}{15}$	$L \times 4 \times \frac{15}{15}$	$L \times 4 \times \frac{15}{15}$	$L \times 4 \times \frac{15}{15}$
5	$L \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 3 \times \frac{15}{15}$	$L \times 4 \times \frac{15}{15}$	$L \times 5 \times \frac{15}{15}$	$L \times 5 \times \frac{15}{15}$	$L \times 5 \times \frac{15}{15}$
6	$L \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 3 \times \frac{15}{15}$	$L \times 4 \times \frac{15}{15}$	$L \times 5 \times \frac{15}{15}$	$L \times 6 \times \frac{15}{15}$	$L \times 6 \times \frac{15}{15}$
7	$L \times \frac{15}{15}$	$L \times 2 \times \frac{15}{15}$	$L \times 3 \times \frac{15}{15}$	$L \times 4 \times \frac{15}{15}$	$L \times 5 \times \frac{15}{15}$	$L \times 6 \times \frac{15}{15}$	$L \times 7 \times \frac{15}{15}$

Sub-totals underlined

Acknowledgement

The authors are very grateful for the encouragement and aid given them by Professor Morrison. They are especially indebted to him for his translations of the various German articles referred to in this thesis.

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